Please read each of the following instructions carefully before attempting questions:

There are EIGHT questions divided in Two Sections and printed both in HINDI and in ENGLISH. Candidate has to attempt FIVE questions in all.

Question Nos. 1 and 5 are compulsory and out of the remaining, THREE are to be attempted choosing at least ONE from each Section.

The number of marks carried by a question/part is indicated against it.

Answers must be written in the medium authorized in the Admission Certificate which must be stated clearly on the cover of this Question-cum-Answer (QCA) Booklet in the space provided. No marks will be given for answers written in a medium other than the authorized one.

Assume suitable data, if considered necessary, and indicate the same clearly.

Unless and otherwise indicated, symbols and notations carry their usual standard meaning.

Attempts of questions shall be counted in sequential order. Unless struck off, attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question-cum-Answer Booklet must be clearly struck off.
SECTION—A

Q. 1(a) Let the probability that a family has exactly \( n \) children be \( \alpha p^n (\alpha > 0, 0 < p < 1) \). Assume that a child can be a boy or a girl with equal probability.

(i) Obtain the probability that a family has exactly \( k \) (\( k \geq 1 \)) boys.

(ii) Given that a family has at least one boy, what is the probability that there are two or more children in that family?

Q. 1(b) Define convergence in distribution and in mean. Let \( \{X_n\}_{n=1}^{\infty} \) be a sequence of independent random variables such that

\[
X_n = n^3, \quad \frac{1}{n^2} \quad \text{with probability} \quad \frac{1}{n^2}
\]

\[
= 0, \quad 1 - \frac{1}{n^2} \quad \text{with probability} \quad 1 - \frac{1}{n^2}
\]

Show that \( X_n \rightarrow 0 \) in distribution.

Q. 1(c) Define a sufficient statistic.

If \( X_1 \) and \( X_2 \) are Bernoulli (p) random variables, examine the sufficiency of \( T_1 \) and \( T_2 \) for \( p \) where \( T_1 = X_1 + X_2 \) and \( T_2 = X_1 + 5X_2 \).
Q. 1(d) Let \( X_1, X_2, \ldots, X_m \) and \( Y_1, Y_2, \ldots, Y_n \) be independent random samples from \( N(3, \sigma_1^2) \) and \( N(2, \sigma_2^2) \) respectively. Find a confidence Interval for \( \frac{\sigma_1^2}{\sigma_2^2} \) at confidence level \( 1 - \alpha \).

Q. 1(e) Consider \( n \) observations \( X_1, X_2, \ldots, X_n \) from \( B(1, \theta) \). Obtain the Bayes estimator for \( \theta \) under quadratic loss function when a conjugate prior is assumed for \( \theta \).

Q. 2(a) How can you find out the discontinuity points of distribution of a random variable if the characteristic function \( \phi(t) \) is given to you? Hence find the c.d.f. of a random variable \( X \) whose characteristic function is \( \frac{1}{4}(1+e^{it}+2e^{2it}) \).

Q. 2(b) Let \( X_n \) be a Poisson \( P(n\theta) \), \( n \geq 1, \theta > 0 \). Use the Central Limit Theorem to find the smallest \( n \) such that

\[
P\left( \frac{X_n - \theta}{\sqrt{\theta}} \leq \frac{\sqrt{\theta}}{10} \right) \geq .99.
\]

Q. 2(c) Let \( X_1, X_2, \ldots, X_{10} \) be independent random samples from \( N(3, \sigma_1^2) \) and \( N(2, \sigma_2^2) \) respectively.

Let \( X_1, X_2, \ldots, X_{10} \) be independent random samples from \( N(3, \sigma_1^2) \) and \( N(2, \sigma_2^2) \) respectively. For the above data find the confidence interval for \( \frac{\sigma_1^2}{\sigma_2^2} \) at confidence level \( 1 - \alpha \).
The distribution under the null hypothesis of $X$ is uniform over 1, 2, ..., 10 and under the alternative hypothesis, the distribution is given by $P\{X = k\} = 0$, for $k = 1, 2, 3, 4$, $P\{X = k\} = \frac{1}{8}$ for $k = 5, 6, 7, 8, 9$ and $P\{X = 10\} = \frac{3}{8}$. Obtain most powerful test of size 0.15 and find its power, based on a sample of size one.

Q. 3(a) एक अभावमान में प्रायिकता $p_1, p_2$ और $p_3$ ($p_1 + p_2 + p_3 = 1$) के साथ तीन सम्भावित परिणामों में से कोई एक परिणाम आ सकता है। $H_0 : p_1 = p_2 = \frac{1}{3}$ की वैकल्पिक कि वे प्रायिकता एक से भिन्न है, के विन्दु रीति के लिए संभावित अनुपात परिणाम का निर्णय कीजिए। परीक्षा के $n$ जांचों (Trials) पर आधारित होकर निर्मित करने की जरूरत है।

A trial can result in one of three possible outcomes with probabilities $p_1$, $p_2$ and $p_3$ ($p_1 + p_2 + p_3 = 1$) respectively. Construct the Likelihood Ratio test of $H_0 : p_1 = p_2 = \frac{1}{3}$ against the alternative that these probabilities are different from $\frac{1}{3}$. The test needs to be constructed on the basis of $n$-trials.

Q. 3(b) आवृत्तियों आकलकों को कोई तैयार करता है?

यदि $X$ का श्रेणी-एच, निम्नलिखित हो,

$$f_X(x) = \frac{x}{\theta} e^{-\frac{x}{\theta}}, \quad 0 < x < \infty, \quad \theta > 0$$

और 50 तत्वों के आकल ने प्रतियोगी

$$\sum_{i=1}^{50} x_i = 8200 \quad \text{और} \quad \sum_{i=1}^{50} x_i^2 = 20,00,000$$

देता है, तो $\theta$ और $\beta$ का आवृत्ति आकलकों को प्राप्त कीजिए।

How does one obtain moment estimators? If $X$ has the pdf

$$f_X(x) = \frac{x}{\theta} e^{-\frac{x}{\theta}}, \quad 0 < x < \infty, \quad \theta > 0$$

and a sample of size 50 yields

$$\sum_{i=1}^{50} x_i = 8200 \quad \text{and} \quad \sum_{i=1}^{50} x_i^2 = 20,00,000$$

find moment estimators of $\theta$ and $\beta$. 15
Q. 3(c) \( H_0 : f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}, -\infty < x < \infty \)

\( H_1 : f(x) = \frac{1}{2} e^{-|x|}, -\infty < x < \infty \)

की

खे विषय परीक्षण के लिए सर्वोच्च शक्तिमान परीक्षण को \( n \) आपात के वादूविच्छेद प्रवश्न के आधार पर प्राप्त कीजिए। यहाँ \( f(x) \) वादूविच्छेद चर \( X \) का पी.डी.एफ. है।

Find a Most Powerful test for testing

\( H_0 : f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}, -\infty < x < \infty \)

versus

\( H_1 : f(x) = \frac{1}{2} e^{-|x|}, -\infty < x < \infty \)

on the basis of a random sample of size \( n \). Here \( f(x) \) is the pdf of the random variable \( X \).

Q. 4(a) माना कि \( F(x) \)

\[
F(x) = \begin{cases} 
0 & \text{if } x < 1 \\
\frac{1}{2} + \frac{(x-1)}{4} & \text{if } 1 \leq x < 2 \\
\frac{7}{8} + \frac{(x-2)}{8} & \text{if } 2 \leq x < 3 \\
1 & \text{if } x \geq 3
\end{cases}
\]

ह्या दिया गया संच्य बंटन फलन (स.ब.फ.) है। \( F \) के असांतत्य बिनुआ के समुच्चय को निकालिए और \( F \) की उनके विकल्प और संंतत भागों के मिश्रण के रूप में व्यक्त कीजिए।

Let \( F(x) \) be the cumulative distribution function (cdf) given by

\[
F(x) = \begin{cases} 
0 & \text{if } x < 1 \\
\frac{1}{2} + \frac{(x-1)}{4} & \text{if } 1 \leq x < 2 \\
\frac{7}{8} + \frac{(x-2)}{8} & \text{if } 2 \leq x < 3 \\
1 & \text{if } x \geq 3
\end{cases}
\]

Find the set of discontinuity points of \( F \) and express \( F \) as a mixture of its discrete and continuous parts.
Q. 4(b) Suppose that X and Y have cdf F and G respectively. Given independent random samples \((x_1, x_2, \ldots, x_m)\) and \((y_1, y_2, \ldots, y_n)\) from these distributions, construct Mann-Whitney test of \(H_0 : F(t) = G(t)\) for all \(t\) against the alternative that \(F(t) > G(t)\) for at least one \(t\). Also indicate the test when the alternative is \(F(t) \neq G(t)\) for some\( t\). State the test you would use when \(m\) and \(n\) are large.

Q. 4(c) Consider observations from Poisson distribution with parameter \(\lambda\). Develop a Sequential Probability Ratio Test (SPRT) to test \(H_0 : \lambda = \lambda_0\) against \(H_1 : \lambda = k\lambda_0, k > 1\). Obtain OC and ASN functions.

SECTION—B

Q. 5(a) A random sample of size 4 from a bivariate normal population provides the following statistics

\[
\bar{x} = \begin{pmatrix} 4 \\ 6 \end{pmatrix},\quad s = \begin{pmatrix} 6 & -2 \\ -2 & 3 \end{pmatrix}
\]

where \(\bar{x}\) is the sample mean and \(s\) is an unbiased estimator of the population dispersion matrix.

Test the hypothesis \(H_0 : \mu = (5,5)'\) where \(\mu\) is the population mean vector.
Q. 5(b) Let $X = (X_1, X_2, \ldots, X_p)'$ be a $p$-dimensional random vector with $E(X) = 0$, $V(X) = \Sigma$. Define partial correlation coefficient $\rho_{12,(3\ldots,p)}$ between $X_1, X_2$. Show that

$$\rho_{12,(3\ldots,p)} = \frac{(\sigma_{12})^2}{\sigma_{11} \sigma_{22}}$$

where $\sigma_{ij}$ is the $(i, j)^{th}$ element of $\Sigma^{-1}$.

Q. 5(c) Consider three independent random variables $Y_1, Y_2$ and $Y_3$ having common variance $\sigma^2$ and $E(Y_1) = \beta_1 + \beta_3, E(Y_2) = \beta_1 + \beta_2, E(Y_3) = \beta, \beta_2, \beta_3$. Determine the condition of estimability of the linear parametric function $\beta$. Obtain a solution of the normal equation and the S.S. due to error.

Q. 5(d) Show that

$$\text{S.N.} = \sum_{h} \left( N_h (\bar{Y}_h - \bar{Y})^2 - \frac{1}{N} \sum_{h} (N - N_h) S_h^2 \right)$$

where $V_{\text{ran}}$ and $V_{\text{prop}}$ are respectively the variances of the estimated means under simple random sampling and stratified random sampling with proportional allocation. All other notations have their usual interpretations.
Q. 5(e) Describe the layout of a $3^3$ experiment in 4 replicates (with 3 blocks per replicate) using complete confounding.

Q. 6(a) For an arbitrary fixed effective size sampling design with positive second order inclusion probabilities, derive the Yates-Grundy form of the variance of the Horvitz-Thompson estimator of a finite population total and hence obtain the Yates-Grundy unbiased estimator of this variance.

Q. 6(b) Let $X = (X_1, X_2, X_3)'$ have the joint moment generating function (mgf)

$$M_X(t) = \exp \left[ t_1 - t_2 + 2t_3 + \frac{t_1^2}{2} + 2t_3^2 - \frac{t_1 t_2}{2} - t_1 t_3 \right]$$

where $t = (t_1, t_2, t_3)'$. Then:

(i) Obtain the covariance matrix and the mean vector of $X$.
(ii) Find a constant $C$ such that $P(2X_1 - 3X_2 + X_3 > C) = 0.95$.
(iii) Derive the conditional distribution of $X_1$ given $X_2 = x_2$ and $X_3 = x_3$.

(If necessary, you can use $P(\tau > 1.645) = 0.05$ and $P(\tau > 1.96) = 0.01$ where $\tau$ is a standard normal variate)
Q. 6(c) For the Gauss Markov Model \((Y, X\beta, \sigma^2I)\), the estimator \(t'Y\) is the best linear unbiased estimator (BLUE) for \(E(t'Y)\) iff \(t'Y\) is uncorrelated with all unbiased estimators of zero.

Q. 7(a) Define a balanced incomplete block design (BIBD). Carry out its intrablock analysis.

Q. 7(b) Suppose a random vector \(X\) has the covariance matrix \(\Sigma = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 5 \end{pmatrix}\). Find the principal components of \(X\) and obtain the proportion of the total variance accounted for by the first two principal components.

Q. 7(c) Consider the simple linear regression model \(y_i = \beta_0 + \beta_1x_i + e_i\), \(i = 1, 2, \ldots, n\). Show that if the \(x_i\)'s are equally spaced (i.e. \(x_i = u + iv\) for fixed values of \(u\) and \(v\)), then \(y_i = y_0 + y_1i + e_i\) is an equivalent reparametrization (in the sense that both the design matrices have the same column space).

Q. 6(c) गौस-मार्कोव मॉडल \((Y, X\beta, \sigma^2I)\) के लिए, \(t'Y\) \(E(t'Y)\) का सर्वोत्तम-रैखिक-अनभिनत अक्लक (BLUE) केवल और केवल तभी है जब \(t'Y\) शून्य के सभी अनभिनत अक्लकों से अ-सहसंबंधित है।

Q. 7(a) संतुलित अनुपात लंबक हिजाइन (की,आई,की,डी.) की परिभाषा दीजिए। इसके अंत:लंबक विश्लेषण को पूरा कीजिए।

Q. 7(b) मानिए कि एक पारदर्शिक सदिश \(X\) का संह्यास आयूर \(\Sigma = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 5 \end{pmatrix}\) है। \(X\) के प्रथम घटकों की मात्रा कीजिए और प्रथम दो प्रथम घटकों के कारण कुल प्रमाण का अंश ग्रान्त कीजिए।

Q. 7(c) विचारिए साधारण रैखिक मॉडल \(y_i = \beta_0 + \beta_1x_i + e_i\), \(i = 1, 2, \ldots, n\). दर्शाइए कि यदि \(x_i\) समान दूरी पर स्थित हों (अर्थात \(x_i = u + iv\), \(u\) और \(v\) के निर्देश मानों के लिए), तो \(y_i = y_0 + y_1i + e_i\) एक समतुल्य पुनर्चकृतकरण है। दोनों हिजाइन आयूर का एक ही स्तर दिक्कत है के अर्थ में.)

Consider the simple linear regression model \(y_i = \beta_0 + \beta_1x_i + e_i\), \(i = 1, 2, \ldots, n\). Show that if the \(x_i\)'s are equally spaced (i.e. \(x_i = u + iv\) for fixed values of \(u\) and \(v\)), then \(y_i = y_0 + y_1i + e_i\) is an equivalent reparametrization (in the sense that both the design matrices have the same column space).
Q. 8(a) Consider a random sample \( X_1, X_2, \ldots, X_n \) from a \( p \)-dimensional normal population with mean vector \( \mu \) and dispersion matrix \( \Sigma \). The purpose of the problem is to construct the likelihood ratio test (LRT) for testing \( H_0 : \Sigma = \sigma^2 I_p \). Find the maximum likelihood estimate (MLE) of \( \mu \) and \( \sigma^2 \). Write down the expression for \(-2\log_e \lambda\) in terms of \( p, n, \overline{X} \) and the elements of the sample covariance matrix \( S \).

Q. 8(b) What is meant by confounding in a factorial experiment? Why is confounding used even at the cost of loss of information on the confounded effects? Explain the terms ‘complete confounding’ and ‘partial confounding’.

Q. 8(c) A simple random sample of \( n \) clusters, each containing \( M \) elements, is drawn from the \( N \) clusters of the population and the clusters sampled are enumerated completely. Suggest an unbiased estimator of the population mean per element and derive the expression for the variance of the proposed estimator in terms of the population intraclass correlation coefficient.